# $E_0$ and Dynamics of Polish Group Actions

**Francis Adams**

## 1 Introduction

Given a group $G$ acting on a set $X$, there is a natural equivalence relation $E_G^X$ on $X$ where two elements of $X$ are equivalent if they are in the same orbit. A framework for investigating the complexity of equivalence relation has been developed in terms of descriptive set theory. Orbit equivalence relations have a prominent place in this framework. This paper will look at some results relating the complexity of the equivalence relation to the dynamical properties of the group action.

## 2 Descriptive Set Theory and Borel Reducibility

Descriptive set theory is the study of definable subsets of Polish spaces.

**Definition 2.1.** *A Polish space is a separable, completely metrizable topological space.*

**Example 1.** *Examples of Polish spaces*

1. *Any countable discrete space.*
2. $\mathbb{R}, \mathbb{R}^n$, *and* $\mathbb{R}^\omega$ *(the set of real valued sequences).*
3. $2^\omega$, *the space of all one-way infinite binary sequences, with the metric* $d(x,y) = 2^{-min\{n:\, x_n \neq y_n\}}$
4. *A subspace of a Polish space is Polish iff it is* $G_\delta$.
5. *The countable product of Polish spaces is Polish.*

The following definition gives examples of what is meant by 'definable' subsets.

**Definition 2.2.** *Let $X, Y$ be Polish spaces.*

- *A set $A \subseteq X$ is Borel if it is in the $\sigma$-algebra generated by the open sets of $X$.*
- *A function $f : X \to Y$ is Borel if the inverse image of every open set is Borel.*
- *A set $A \subseteq X$ is analytic if it is the continuous image of a Borel subset of a Polish space.*

By a theorem of Suslin, a set is Borel iff both it and its complement are analytic. There do exist sets which are analytic but not Borel, and of course there are subsets of Polish spaces which are not analytic.

Say that an equivalence relation $E$ on a Polish space $X$ is Borel (analytic) if it is Borel (analytic) as a subset of $X \times X$. Such definable equivalence relations are compared to each other via Borel reducibility.

**Definition 2.3.** *Let $E, F$ be equivalence relations on Polish spaces $X, Y$. A reduction of $E$ to $F$ is a function $f : X \to Y$ such that $x_1 E x_2 \iff f(x_1) F f(x_2)$ for all $x_1, x_2 \in X$. Say that $E$ is Borel reducible to $F$, written $E \leq_B F$, if there is a Borel reduction $f : X \to Y$. Say $E, F$ are Borel bireducible to each other, written $\sim_B$, if $E \leq_B F$ and $F \leq_B E$. Finally, say that $E$ continuously embeds into $F$, written $E \sqsubseteq_c F$ if there is a continuous, injective reduction $f : X \to Y$.*

**Example 2.** *Examples of definable equivalence relations.*

- *For a Polish space $X$, $id(X)$ is the identity (or equality) relation on $X$. If $X, Y$ are any uncountable Polish spaces, then $id(X) \sim_B id(Y)$. If $E \leq_B id(2^\omega)$, say that the equivalence relation $E$ is smooth.*
- *Define $E_0$ on $2^\omega$ by $x E_0 y \iff \exists m \forall n \geq m[x(n) = y(n)]$. So $E_0$ is eventual agreement of infinite binary sequences.*
- *Define $E_1$ on $\mathbb{R}^\omega$ by $x E_1 \Leftrightarrow \exists m \forall n \geq m[x(n) = y(n)]$.*

If $E$ is an equivalence relation on $X$, for $x \in X$ let $[x]_E$ be its equivalence class, or $[x]$ if the equivalence relation is clear from context.

A basic theorem for understanding subsets of product spaces is the Kuratowski-Ulam theorem. Recall that a subset of a topological space $A \subset X$ has the Baire property if there is an open subset $U$ of $X$ such that the symmetric difference $A \triangle U$ is meager in $X$. All analytic sets, and hence all Borel sets, have the Baire property.

**Theorem 2.4.** *Let $X, Y$ be second countable and $A \subseteq X \times Y$ have the Baire property. Then $A$ is meager iff $\{x \in X : \{y \in Y : (x, y) \in A\}$ is meager $\}$ is comeager (i.e. comeagerly many $x$ have meager sections $A_x$).*

# 3 Orbit Equivalence Relations

Given a Polish group $G$ and a Polish space $X$, say that $X$ is a Polish $G$-space if $G$ acts continuously on $X$. Such an action induces an equivalence relation on $X$:

**Definition 3.1.** *Let $X$ be a Polish $G$-space. Define the equivalence relation $E_G^X$ on $X$ by relating $x, y \in X$ iff there is a $g \in G$ so $g \cdot x = y$. More generally, if $G$ is just a group of homeomorphisms of $X$, then $E_G$ is the equivalence relation relating $x, y \in X$ iff there is a $g \in G$ so $g(x) = y$.*

Every orbit equivalence relation induced by a Polish action is analytic, since it is the projection onto $X^2$ of the closed set set

$$\{(g, x, y) : g \cdot x = y\} \subseteq G \times X^2$$

$E_0$ can be viewed as an orbit equivalence relation. Let the direct sum $\oplus_{n \in \omega} \mathbb{Z}_2$ act on $2^\omega$ by coordinatewise addition mod 2. Since each element of the direct sum has finite support, acting on $x \in 2^\omega$ by an element $g$ only changes finitely many values. So $x, g \cdot x$ are $E_0$-equivalent.

By a result of Kechris and Louveau, $E_1$ is not Borel reducible to $E_G^X$ for any Polish group $G$ and Polish $G$-space $X$.

We state a theorem of Effros for later use:

**Theorem 3.2.** *Let $G$ be a Polish group and $X$ be a Polish $G$-space. Then the following are equivalent for each $x \in X$:*

*i) $G \cdot x$ is non meager in its relative topology.*

*ii) $G \cdot x$ is $G_\delta$ in $X$.*

# 4 $E_0$

By the Glimm-Effros Dichotomy below, $E_0$ is the first Borel equivalence relation above the identity in the quasi-ordering of equivalence relations under Borel reducibility. Here we will look more closely at $E_0$, eventually proving that is it not smooth.

We start by identifying a property that leads to non-smoothness.

**Definition 4.1.** *Let $X$ be Polish with $E$ an equivalence relation on $X$. Say that $E$ is generically ergodic if every $E$-invariant Borel set is meager or comeager.*

**Proposition 4.2.** *Let $G$ be a group of homeomorphisms on a Polish space $X$. The following are equivalent:*

1. *$E_G$ is generically ergodic.*
2. *Every nonempty $E_G$-invariant open set is dense.*
3. *There is an invariant dense $G_\delta$ set $Y \subseteq X$ all of whose orbits are dense in $X$.*
4. *There exists a dense orbit.*
5. *Every invariant set with the Baire property is meager or com eager.*

Because $E_0$-orbits are closed under finite changes to any element, every orbit intersects each basic open set of $2^\omega$. So since $E_0$ is an orbit equivalence relation of a Polish group action with all orbits dense, it is generically ergodic.

We can also note that $E_0$ has no comeager orbits, since each orbit is countable. This fact implies that $E_0$ has no $G_\delta$ orbits, for such an orbit would be dense $G_\delta$, hence comeager. More importantly, the absence of comeager orbits along with the generic ergodicity implies that $E_0$ is not smooth.

**Theorem 4.3.** *Let $E$ be an equivalence relation on a Polish space $X$. If $E$ is generically ergodic and has no comeager orbits, then $E$ is not smooth.*

*Proof.* We will use an alternative characterization of smoothness. Let $E$ be an equivalence relation on $X$ and $\mathcal{S}$ a family of Borel subsets of $X$. Say that $\mathcal{S}$ is a Borel generating family for $E$ if for any $x, y \in X$,

$$xEy \Leftrightarrow \text{ for any } S \in \mathcal{S}(x \in S \leftrightarrow y \in S)$$

It is not hard to show that an equivalence relation is smooth iff it has a countable Borel generating family.

Now suppose that our $E$ is smooth with countable Borel generating family $\{S_n\}_{n\in\omega}$. Each $S_n$ is $E$-invariant, so they are meager or comeager. Let

$$C = \bigcap\{X \setminus S_n : S_n \text{ is meager }\} \cap \bigcap\{S_n : S_n \text{ is comeager }\}$$

As the intersection of countably many comeager sets, $C$ is comeager. But, $C$ is in fact an equivalence class. If $x, y \in C$, they are in exactly the same $S_n$, hence $xEy$. Conversely, if $x \in C$ and $xEy$, then $x, y$ are in exactly the same $S_n$ hence $y \in C$ as well. So $C$ is a comeager equivalence class, a contradiction. $\square$

# 5 Glimm-Effros Dichotomy

In modern terminology, the Glimm-Effros dichotomy refers to the following theorem of Harrington, Kechris, and Louveau:

**Theorem 5.1.** *Let $E$ be a Borel equivalence relation on a Polish space $X$. Then either $E$ is smooth or $E_0 \sqsubseteq_c E$.*

Our aim here is to prove a modern restatement of the original Glimm-Effros dichotomy theorem, a much weaker result concerning orbit equivalence relations. We begin with some preliminary results.

**Theorem 5.2.** *(Becker-Kechris) Let $X$ be Polish and $G$ a group of homeomorphisms of $X$. Suppose that $E_G$ is meager and that there is a dense orbit. Then $E_0 \sqsubseteq_c E_G$.*

*Proof.* Let $W_0 \supseteq W_1 \supseteq W_2 \supseteq \cdots$ be dense open subsets of $X^2$ so that $E_G \cap \bigcap W_n = \emptyset$. Since the diagonal is a closed nowhere dense set, we may assume that $W_0 \cap \{(x,x) : x \in X\} = \emptyset$. The proof relies on the following construction, which we assume without proof:

Let $2^{<\omega}$ be the set of all finite binary strings. Let $\{U_s\}_{s\in 2^{<\omega}}$ be open subsets of $X$ and $g_{s,t} \in G$ for $s,t \in 2^{<\omega}$ with $|s| = |t|$ be such that for all $s,t,u \in 2^{<\omega}$ with $|s| = |t| = |u|$ :

a) $U_\emptyset = X, diam(U_s) < 2^{-|s|}, \overline{U_{s^\frown 0}}, \overline{U_{s^\frown 1}} \subseteq U_s, U_{s^\frown 0} \cap U_{s^\frown 1} = \emptyset$

b) If $|s| = |t| = n$ and $s(n-1) \neq t(n-1)$, then $U_s \times U_t \subseteq W_n$

c) $g_{s,t} \cdot U_s = U_t, g_{s,s} = 1_G, g_{t,s} = g_{s,t}^{-1}, g_{s,u} = g_{t,u} g_{s,t}$

d) If $n \leq |t| = |s|$ is the largest integer with $s(n-1) \neq t(n-1)$, then $g_{s,t} = g_{s\restriction n, t\restriction n}$.

Given this construction, for $x \in 2^\omega$ let $f(x)$ be the unique element of $\bigcap_{n=0}^\infty U_{x\restriction n} = \bigcap_{n=0}^\infty \overline{U_{x\restriction n}}$. Since the diameters go to zero, $f(x)$ is well-defined. Since $U_{s^\frown 0} \cap U_{s^\frown 1} = \emptyset$, it is one-to-one. To get $d_X(f(x), f(y) < \frac{1}{N}$ we just need $x,y \in 2^\omega$ to agree on the first $N$ entries, so $f$ is continuous.

It remains to show that $f$ is a reduction of $E_0$ to $E_G$. Suppose that $x \not{E}_0 y$. Then for infinitely many $n$, we have $x(n-1) \neq y(n-1)$. By c) this means that for such $n$, $U_{x\restriction n} \times U_{y\restriction n} \subset W_n$. So $(f(x), f(y)) \in \bigcap W_n$, hence $f(x) \not{E}_G f(y)$. If $x E_0 y$, let $n$ be the largest integer so $x(n-1) \neq y(n-1)$. Let $g = g_{x\restriction n, y\restriction n}$. By d) if $n \leq m$, then also $g = g_{x\restriction m, y\restriction m}$. So $g \cdot U_{x\restriction m} = U_{y\restriction m}$ for all $m \geq n$. Hence $g \cdot f(x) = f(y)$, i.e. $f(x) E_0 f(y)$.

$\square$

**Corollary 5.3.** *Let $G$ be a Polish group and $X$ a Polish $G$-space. Suppose the action has no $G_\delta$ orbit. Then $E_0 \sqsubseteq_c E_G^X$.*

*Proof.* Let $x \in X$ and $Y = \overline{[x]}$. Then $Y$ is a Polish $G$-space under the inherited action. Also $E_G^Y \sqsubseteq_c E_G^X$ by the identity embedding. Since $X$ has no $G_\delta$ orbits neither does $Y$. Then by Effros' theorem above every orbit is meager in itself, so also every orbit of $Y$ is meager in $Y$. Since $E_G^Y$ is analytic, it has the Baire property. Because every section is meager, $E_G^Y$ is meager by the Kuratowski-Ulam theorem. Since $Y$ obviously has a dense orbit, we have by the previous theorem $E_0 \sqsubseteq_c E_G^Y$, and hence $E_0 \sqsubseteq_c E_G^X$. □

Now we prove a dichotomy theorem that will lead to the results of Effros and Glimm.

**Theorem 5.4.** *Let $G$ be a Polish group and $X$ a Polish $G$-space. Suppose that every $G_\delta$ orbit is $F_\sigma$. Then either every orbit is $G_\delta$ or $E_0 \sqsubseteq_c E_G^X$. In particular, $E_G^X$ is smooth or $E_0 \sqsubseteq_c E_G^X$.*

*Proof.* Suppose that $E_0 \not\sqsubseteq_c E_G^X$. Let $x \in X$ and let $Y = \{y \in X : \overline{[x]} = \overline{[y]}\}$. Then $Y$ is invariant, and $Y$ is also $G_\delta$. This can be seen by letting $\mathcal{U}$ be a countable base for $X$ and then

$$y \in Y \Leftrightarrow \forall U \in \mathcal{U}(U \cap [x] \neq \emptyset \Rightarrow y \in [U]) \wedge \forall U \in \mathcal{U}(U \cap [x] = \emptyset \Rightarrow y \notin [U])$$

So $Y$ is a Polish $G$-space with the inherited action of $G$. Thus $E_G^Y \sqsubseteq_c E_G^X$ by the identity map, so by our assumption $E_0 \not\sqsubseteq_c E_G^Y$.

By the corollary above, $E_G^Y$ has a $G_\delta$ orbit, say $[y] \subseteq Y$. Also, every orbit of $Y$ is dense. For if $y \in Y$, then $y \in \overline{[y]} = \overline{[x]}$ so $Y \subseteq \overline{[x]}$ and if $y \in \overline{[x]}$ then $\overline{[y]} = \overline{[x]}$ so $y \in Y$. Thus $\overline{Y} = \overline{[x]}$ so too $\overline{Y} = \overline{[y]}$ for every $y \in Y$.

If $Y \setminus [y] \neq \emptyset$ then it is is dense $G_\delta$ since it contains an orbit, which is dense, and $[y]$ is $F_\sigma$. But then it is disjoint from the dense $G_\delta$ set $[y]$, a contradiction. So $Y = [y] = [x]$, and in particular the orbit $[x]$ is $G_\delta$. Because $x$ was arbitrary, every orbit of $E_G^X$ is $G_\delta$.

If every orbit of $E_G^X$ is $G_\delta$, then $E_G^X$ is smooth. This is because the Borel map $x \to \overline{[x]}$ is a reduction, i.e. $xE_G^X y \Leftrightarrow \overline{[x]} = \overline{[y]}$. For suppose $xE_G^X y$. Then $[x] = [y]$ so $\overline{[x]} = \overline{[y]}$. Conversely, if $C = \overline{[x]} = \overline{[y]}$, then both $[x], [y]$ are dense $G_\delta$ in $C$. So they must intersect and must in fact be equal, yielding $xE_G^X y$. □

Now we can present the dichotomies of Glimm and Effros:

**Corollary 5.5.** *(Effros) Let $G$ be Polish, $X$ a Polish $G$-space. Suppose $E_G^X$ is $F_\sigma$. Then either $E_G^X$ is smooth or $E_0 \sqsubseteq_c E_G^X$.*

*Proof.* If $E_G^X$ is $F_\sigma$ then so is every orbit, in particular all $G_\delta$ orbits are $F_\sigma$. □

**Corollary 5.6.** *(Glimm) Let $G$ be a locally compact Polish group and $X$ a Polish $G$-space. Then either $E_G^X$ is smooth or $E_0 \sqsubseteq_c E_G^X$.*

*Proof.* A Polish space is locally compact iff it is $K_\sigma$, that is a union of countably many compact sets. Then $E_G^X$ is the projection onto $X^2$ of the $K_\sigma$ set

$$\{(g, x, y) : g \cdot x = y\} \subseteq G \times X^2$$

so $E_G^X$ is $K_\sigma$ and in particular $F_\sigma$. □

# References


[1] S. Gao, *Invariant Descriptive Set Theory*, (CRC Press, 2009)

[2] A.Kechris, *Classical Descriptive Set Theory*, (Springer, 1995)