Discrete Spectrum Transformations

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1 Introduction

An important problem in ergodic theory is to find classifications of various types of measure preserving transformations. An example of this is the theorem of Ornstein that two Bernoulli shifts are isomorphic iff they have the same entropy. So a Bernoulli shift can be assigned a real number that completely determines its isomorphism type. Can this be done for other types of transformations? Perhaps not with real numbers, but possibly with other types of invariants. This type of problem fits into the framework of definable equivalence relations in descriptive set theory. In this setting, equivalence problems are classified by the invariants that must be assigned to determine an object up to the equivalence is being considered.

2 Eigenvalues and Eigenfunctions

If (X, Σ, μ, f) is a probability measure preserving transformation, there is an associated linear isometry $U_f : L^2(\mu) \to L^2(\mu)$ defined by $U_f(\alpha) = \alpha \circ f$. Eigenfunctions of U_f, or more briefly of f, are $\alpha \neq 0$ such that there is a $\lambda \in \mathbb{C}$ so $U_f(\alpha) = \lambda\alpha$. Say λ is the eigenvalue associated with α.

Proposition 2.1. *Let (X, Σ, μ, f) be an ergodic probability measure preserving transformation.*

1. *If $U_f\alpha = \lambda\alpha$ for nonzero $\alpha \in L^2(\mu)$, then $|\lambda| = 1$ and $|\alpha|$ is constant a.e.*

2. *If $\lambda_1 \neq \lambda_2$ are eigenvalues with associated eigenfunctions α_1, α_2, then $\alpha_1 \perp \alpha_2$*

3. *If λ has eigenfunctions α_1, α_2, there is $c \in \mathbb{C}$ such that $\alpha_1 = c\alpha_2$ a.e.*

4. *The eigenvalues of f form a subgroup of S^1*

Proof. 1. First, $\|\alpha\| = \|U_f \alpha\| = |\lambda|\|\alpha\|$ so $|\lambda| = 1$. Then $|\alpha \circ f| = |U_f(\alpha)| = |\lambda||\alpha| = |\alpha|$ so $|\alpha|$ is f-invariant, hence constant a.e by ergodicity.

2. Suppose $\lambda_1 \neq \lambda_2, U_{f_1}\alpha_1 = \lambda_1\alpha_1$, and $U_{f_2}\alpha_2 = \lambda_2\alpha_2$. Then

$$\langle \alpha_1, \alpha_2 \rangle = \langle U_{f_1}\alpha_1, U_{f_2}\alpha_2 \rangle = \langle \lambda_1\alpha_1, \lambda_2\alpha_2 \rangle = \lambda_1\overline{\lambda_2}\langle \alpha_1, \alpha_2 \rangle$$

Since $\lambda_1 \neq \lambda_2$, $\lambda_1\overline{\lambda_2} \neq 1$, so it must be that $\langle \alpha_1, \alpha_2 \rangle = 0$.

3. Since α_2 isn't the zero function, but $|\alpha_2|$ is constant a.e., we know that $\alpha_2(x) \neq 0$ for a.e. x. Then

$$\frac{\alpha_1}{\alpha_2} \circ f = \frac{\alpha_1 \circ f}{\alpha_2 \circ f} = \frac{\lambda\alpha_1}{\lambda\alpha_2} = \frac{\alpha_1}{\alpha_2}$$

almost everywhere. So by ergodicity there is a constant $c \in \mathbb{C}$ such that $\frac{\alpha_1}{\alpha_2} = c$ a.e.

4. Let λ_1, λ_2 be eigenvalues of f, so $\alpha_1 \circ f = \lambda_1\alpha_1$ and $\alpha_2 \circ f = \lambda_2\alpha_2$ for some nonzero α_1, α_2. Then $\overline{\alpha_2} \circ f = \overline{\lambda_2}\overline{\alpha_2}$ so $(\alpha_1\overline{\alpha_2}) \circ f = \lambda_1\overline{\lambda_2}\alpha_1\overline{\alpha_2}$. Hence $\lambda_1\overline{\lambda_2} = \lambda_1\lambda_2^{-1}$ is an eigenvalue, which means the eigenvalues form a subgroup.

□

3 Conjugacy and Spectral Isomorphism

Let (X, Σ, μ) be a probability measure space. Define an equivalence relation on Σ relating two sets if their symmetric difference has μ-measure 0. Let $\tilde{\Sigma}$ be the set of equivalence classes. This measure algebra is a σ-complete Boolean algebra under complement, union, and intersection where σ-complete means closed under countable unions and intersections. Say that two measure algebras $(\tilde{\Sigma}_i, \mu_i)$ are isomorphic if there is a bijection $\phi : \tilde{\Sigma}_2 \to \tilde{\Sigma}_1$ which is an isomorphism of Boolean algebras and $\mu_1(\phi\tilde{B}) = \mu_2(\tilde{B})$ for all $B \in \Sigma_2$. We will freely confuse a set B with its equivalence class $\tilde{B}$ as well as functions and measures f, μ with the induced functions and measures $\tilde{f}, \tilde{\mu}$ on $\tilde{\Sigma}$.

Definition 3.1. *Let $(X_i, \Sigma_i, \mu_i, f_i)$ be probability measure preserving transformations for $i = 1, 2$. They are*

- *conjugate if there is a measure algebra isomorphism $\phi : (\tilde{\Sigma}_2, \mu_2) \to (\tilde{\Sigma}_1, \mu_1)$ such that $\phi f_2^{-1} = f_1^{-1}\phi$*

- *spectrally isomorphic if there is a Hilbert space isomorphism $W : L^2(\mu_2) \to L^2(\mu_1)$ such that $U_{f_1}W = WU_{f_2}$.*

Proposition 3.2. *If $(X_i, \Sigma_i, \mu_i, f_i)$ are conjugate, then they are spectrally isomorphic.*

Proof. Let $\phi : (\tilde{\Sigma}_2, \mu_2) \to (\tilde{\Sigma}_1, \mu_1)$ be a measure algebra isomorphism such that $\phi f_2^{-1} = f_1^{-1}\phi$. Define $W : L^2(\mu_2) \to L^2(\mu_1)$ by setting $W(\chi_B) = \chi_{\phi(B)}$ and extending to all of $L^2(\mu_2)$ using linearity and taking limits. Then W is an isomorphism, so we just have to show $U_{f_1}W = WU_{f_2}$. For $B \in \Sigma_2$ we have

$$U_{f_1}W(\chi_B) = U_{f_1}\chi_{\phi(B)} = \chi_{\phi(B)} \circ f_1 = \chi_{f_1^{-1}\phi(B)}$$

$$= \chi_{\phi f_2^{-1}(B)} = W(\chi_{f_2^{-1}(B)}) = W(\chi_B \circ f_2) = WU_{f_2}(\chi_B)$$

This property extends to all of $L^2(\mu_2)$. □

There are also conditions under which spectral isomorphism implies conjugacy:

Proposition 3.3. *Let $(X_i, \Sigma_i, \mu_i, f_i)$ be spectrally isomorphic as witnessed by $W : L^2(\mu_2) \to L^2(\mu_1)$. Suppose further that*

- *W, W^{-1} map bounded functions to bounded functions*
- *$W(\alpha\beta) = W(\alpha)W(\beta)$ for all bounded α, β.*

Then $(X_i, \Sigma_i, \mu_i, f_i)$ are conjugate.

4 Discrete Spectrum Transformations

We will investigate a particular class of transformations:

Definition 4.1. *A probability measure preserving transformation (X, Σ, μ, f) has discrete spectrum if there is an orthonormal basis for $L^2(\mu)$ consisting of eigenfunctions of f.*

For the main theorem of this section, we need two lemmas.

Lemma 4.2. *Let (X, Σ, μ) be a probability space. A function $h \in L^2(\mu)$ is bounded iff $hf \in L^2(\mu)$ for all $f \in L^2(\mu)$.*

Lemma 4.3. *Let H be a discrete abelian group and K be a divisible subgroup of H. Then there exists a homomorphism $\phi : H \to K$ such that $\phi \restriction K = id_K$.*

The main theorem, due to Halmos and von Neumann in 1942, says that the set of eigenvalues is a complete invariant for determining conjugacy of discrete spectrum transformations.

Theorem 4.4. *Let $(X_i, \Sigma_i, \mu_i, f_i)$ be ergodic probability measure preserving transformations with discrete spectrum. The following are equivalent:*

1. *f_1, f_2 are spectrally isomorphic.*

2. *f_1, f_2 have the same eigenvalues.*

3. *f_1, f_2 are conjugate.*

Proof. $3 \Rightarrow 1$ is Proposition 3.2.

For $1 \Rightarrow 2$, suppose that f_1, f_2 are spectrally isomorphic as witnessed by some W. Let λ be an eigenvalue of f_2 with eigenfunction α. Then

$$U_{f_1} W(\alpha) = W U_{f_2}(\alpha) = W(\lambda\alpha) = \lambda W(\alpha)$$

so λ is also an eigenvalue of f_1 with eigenfunction $W(\alpha)$. Similarly every eigenvalue of f_1 is an eigenvalue of f_2.

To prove $2 \Rightarrow 1$, for each eigenvalue λ choose $\alpha_\lambda \in L^2(\mu_1)$ and $\beta_\lambda \in L^2(\mu_2)$ such that $|\alpha_\lambda| = |\beta_\lambda| = 1$ and $U_{f_1}\alpha_\lambda = \lambda\alpha_\lambda, U_{f_2}\beta_\lambda = \lambda\beta_\lambda$. Define $W : L^2(\mu_2) \to L^2(\mu_1)$ by $W(\beta_\lambda) = \alpha_\lambda$ and extending linearly using that we are sending a basis to a basis. Then W is a Hilbert space isomorphism and $WU_{f_2} = U_{f_2}W$ holds since it holds for the β_λ.

Now to show $2 \Rightarrow 3$. Let Λ be the group of eigenvalues of f_1 and f_2. For $\lambda \in \Lambda$ find $\alpha_\lambda \in L^2(\mu_1)$ and $\beta_\lambda \in L^2(\mu_2)$ as above. For all $\lambda, \kappa \in \Lambda$ we have $U_{f_1}\alpha_{\lambda\kappa} = \lambda\kappa\alpha_{\lambda\kappa}$ and $U_{f_1}\alpha_\lambda\alpha_\kappa = (\alpha_\lambda \circ f)\cdot(\alpha_\kappa \circ f) = \lambda\kappa\alpha_\lambda\alpha_\kappa$. So by Lemma 2.1 there is $r(\lambda,\kappa) \in S^1$ such that $\alpha_\lambda\alpha_\kappa = r(\lambda,\kappa)\alpha_{\lambda\kappa}$ almost everywhere. We will show that we may assume this constant is 1.

Let $H = \{f : X \to S^1\}$. So H is an abelian group under multiplication. And S^1, viewed as the constant functions, forms a divisible subgroup of H. Let $\phi : H \to S^1$ be a group homomorphism such that $\phi \restriction S^1 = id_{S^1}$. Let $\alpha^*_\lambda = \overline{\phi(\alpha_\lambda)}\alpha_\lambda$. Then $|\alpha^*_\lambda| = 1, U_{f_1}\alpha^*_\lambda = \overline{\phi(\alpha_\lambda)}\alpha_\lambda \circ f = \overline{\phi(\alpha_\lambda)}\lambda\alpha_\lambda = \lambda\alpha^*_\lambda$, and $\{\alpha^*_\lambda : \lambda \in \Lambda\}$ is a basis for $L^2(\mu_1)$. Also,

$$\alpha^*_\lambda\alpha^*_\kappa = \overline{\phi(\alpha_\lambda)}\alpha_\lambda\overline{\phi(\alpha_\kappa)}\alpha_\kappa$$

$$= \overline{\phi(\alpha_\lambda\alpha_\kappa)}\alpha_\lambda\alpha_\kappa = \overline{r(\lambda,\kappa)\phi(\alpha_{\lambda\kappa})}r(\lambda,\kappa)\alpha_{\lambda\kappa}$$

$$= \overline{\phi(\alpha_{\lambda\kappa})}\alpha_{\lambda\kappa} = \alpha^*_{\lambda\kappa}$$

So we may assume $\alpha_\lambda\alpha_\kappa = \alpha_{\lambda\kappa}$ and similarly for the β_λ.

Define $W : L^2(\mu_2) \to L^2(\mu_1)$ by $W(\beta_\lambda) = \alpha_\lambda$ and extending linearly. This W is a Hilbert space isomorphism and $WU_{f_2} = U_{f_1}W$. So it suffices to show W satisfies the conditions from Lemma 3.3. We can get $W(\gamma)W(\delta) = W(\gamma\delta)$ using the continuity of W and the fact that it holds on a basis. Finally, assuming δ is bounded, we have $\gamma\delta \in L^2(\mu_2)$ for all $\gamma \in L^2(\mu_2)$ so $W(\gamma\delta) = W(\gamma)W(\delta) \in L^2(\mu_1)$ for all $\gamma \in L^2(\mu_2)$. But since W is an isomorphism, this means $W(\delta)\theta \in L^2(\mu_1)$ for all $\theta \in L^2(\mu_1$. Hence $W(\delta)$ is bounded, and W sends bounded functions to bounded functions. Similarly this holds for W^{-1} and the proof is complete. $\square$

5 Descriptive Set Theory and Borel Reducibility

Descriptive set theory is the study of definable subsets of Polish spaces.

Definition 5.1. *A Polish space is a separable, completely metrizable topological space.*

Example 1. *Examples of Polish spaces*

1. *Any G_δ subset of a Polish space (in particular any closed subset).*
2. *$\mathbb{C}, \mathbb{C}^n$, and $\mathbb{C}^\mathbb{N}$ (the set of complex valued sequences).*
3. *$2^\mathbb{N}$ and $\mathbb{N}^\mathbb{N}$ with the metric $d(x, y) = 2^{-min\{n : x_n \neq y_n\}}$*
4. *Any separable Banach space.*

The following definition gives examples of what is meant by 'definable' subsets.

Definition 5.2. *Let X, Y be Polish spaces.*

- *A set $A \subseteq X$ is Borel if it is in the σ-algebra generated by the open sets of X.*
- *A function $f : X \to Y$ is Borel if the inverse image of every open set is Borel.*
- *A set $A \subseteq X$ is analytic if it is the continuous image of a Borel subset of a Polish space.*
- *A set $A \subseteq X$ is coanalytic if it is the complement of an analytic set.*

By a theorem of Suslin, a set is Borel iff it is both analytic and coanalytic. There do exist sets which are analytic or coanalytic but not Borel, and of course there are subsets of Polish spaces which are neither analytic nor coanalytic.

Say that an equivalence relation E on a Polish space X is Borel (analytic, coanalytic) if it is Borel (analytic, coanalytic) as a subset of $X \times X$. Such definable equivalence relations are compared to each other via Borel reducibility.

Definition 5.3. *Let E, F be equivalence relations on Polish spaces X, Y. Say that E is Borel reducible to F, written $E \leq_B F$, if there is a Borel function $f : X \to Y$ such that $x_1 E x_2 \iff f(x_1) F f(x_2)$ for all $x_1, x_2 \in X$. Write $E <_B F$ if $E \leq_B F$ but not $F \leq_B E$. Say E, F are Borel bireducible to each other, written $\sim_B$, if $E \leq_B F$ and $F \leq_B E$.*

Example 2. *Examples of definable equivalence relations.*

- *For a Polish space X, $id(X)$ is the identity (or equality) relation on X. For brevity we may suppress X. If X, Y are any uncountable Polish spaces, then $id(X) \sim_B id(Y)$. The Silver Dichotomy Theorem says that for a coanalytic equivalence relation E on a Polish space X, either E only has countably many equivalence classes, or $id(X) \leq_B E$. An equivalence relation E is said to be smooth if $E \leq_B id$. Ornstein's theorem mentioned in the introduction says that isomorphism of Bernoulli shifts is smooth, as witnessed by sending a transformation to its entropy.*

- *Let an equivalence relation be called countable if each of its equivalence classes is countable. There is an equivalence relation E_∞ which is universal for countable equivalence relations; i.e. E_∞ is countable and if E is another countable equivalence relation, then $E \leq_B E_\infty$.*

- *Let E be an equivalence relation on a Polish space X. Define an equivalence relation E^+ on $X^{\mathbb{N}}$, called the jump of E, by*

$$(x_n) E^+ (y_n) \Leftrightarrow \{[x_n] : n \in \mathbb{N}\} = \{[y_n] : n \in \mathbb{N}\}$$

  That is, the sequences (x_n) and (y_n) have representatives from exactly the same set of equivalence classes. It is easy to see that for any equivalence relation $E \leq_B E^+$, and by a theorem of Friedman and Stanley, $E <_B E^+$ for any Borel E with more than one equivalence class. In particular we will be interested in $id(X)^+$ on $X^{\mathbb{N}}$. Here $(x_n) id^+ (y_n) \Leftrightarrow \{x_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\}$, so the two sequences have the same range. It can be shown that $E_\infty^+ \sim_B id^+$.

6 Descriptive Complexity of Conjugacy of Discrete Spectrum Transformations

We now investigate the Halmos von-Neumann theorem within the framework of Borel reducibility of definable equivalence relations. We start with some preliminary work showing that the conjugacy of discrete spectrum transformations is in fact an equivalence relation on a Polish space.

For analyzing complexity, we will just be looking at Lebesgue spaces, i.e. probability measure spaces (X, Σ, μ) which are isomorphic to $[0, 1]$ with Lebesgue measure on the Lebesgue measureable sets. Here isomorphic means there is an invertible measure-preserving transformation between sets of full measure.

For a Lebesgue space (X, Σ, μ), $L^2(\mu)$ is a separable, countably-infinte dimensional Hilbert space. So the unitary group $U(L^2(\mu))$ with the weak topology (which coincides with the strong topology) is a Polish group. The subset of $U(L^2(\mu))$ which are U_f for some measure preserving transformation f forms a G_δ subset, so they form a Polish space. Then finally, the correspondence $f \to U_f$ induces a Polish topology on the space of measure preserving transformations. Call this Polish space $\mathcal{M}$. It can also be shown that the ergodic measure preserving transformations form a dense G_δ subset of $\mathcal{M}$.

Let $\mathcal{D} \subseteq \mathcal{M}$ be the collection of ergodic discrete spectrum transformations. This set is Borel: for every $\alpha < \omega_1$ the measure distal transformations with norm $\leq \alpha$ is a Borel set, and the discrete spectrum transformations are the measure distal transformations with norm 0. It is also worth noting that the set of all measure distal transformations is coanalytic, non-Borel.

So conjugacy of discrete spectrum transformations is an equivalence relation on a Polish space X. Before the main theorem, we need two more results.

First, a classical uniformization theorem. A uniformization of $P \subset X \times Y$ is a function $B \subseteq P$ such that $proj_X(B) = proj_X(P)$, i.e. they have the same projection onto X.

Theorem 6.1 (Lusin-Novikov)**.** *Let X, Y be Polish and let $P \subseteq X \times Y$ have countable vertical sections P_x for each $x \in X$. Then P has a Borel uniformization. Moreover, $P = \bigcup P_n$ where each P_n is a Borel function defined on some subset of X. From this we can also conclude that there is a Borel function $f : X \times \mathbb{N} \to Y$ such that $P_x = \{f(x, n) : n \in \mathbb{N}\}$.*

Second, a complexity calculation.

Lemma 6.2. *$\{(L, \lambda) : \lambda$ is an eigenvalue of $L\} \subseteq U(L^2(\mu)) \times \mathbb{C}$ is Borel*

The map $(L, \lambda) \to L - \lambda I$ is Borel. Since our set is the inverse image under this map of operators with non-trivial kernel, we just have to show the set of such operators is Borel. Indeed, if we fix a basis of $L^2(\mu)$, say $\{e_i : i \in \mathbb{N}\}$, then

$$ker(L) \neq 0$$
$$\Leftrightarrow \exists h \in L^2(\mu)\, h \neq 0 \text{ and } L(h) = 0$$
$$\Leftrightarrow \forall P\, (P \text{ is a projection onto } ker(L) \to \exists i\, P(e_i) \neq 0$$

The second condition says the set of operators with non-trivial kernel is analytic, while the third says it is coanalytic, so by Suslin's theorem it is Borel.

Now we can turn to the main theorem of this section.

Theorem 6.3. *Conjugacy of ergodic discrete spectrum transformations is Borel bi-reducible to id^+.*

Proof. Here we will show just one reduction, finding a Borel function $\phi : \mathcal{D} \to (S^1)^\omega$ reducing conjugacy on $\mathcal{D}$ to id^+ on $(S^1)^\omega$. Let $\phi(f)$ be an enumeration of the eigenvalues of f. Clearly this is a reduction; by Halmos von-Neumann, two transformations $f_1, f_2 \in \mathcal{D}$ are conjugate iff $\phi(f_1)$ and $\phi(f_2)$ enumerate the same set. So we only have to show there exists such a function ϕ which is Borel.

We know that

$$\{(L, \lambda) : \lambda \text{ is an eigenvalue of } L, L = U_f \text{ where } f \text{ has discrete spectrum}\}$$

is a Borel subset of $U(L^2(\mu)) \times \mathbb{C}$. For Lebesgue spaces, $L^2(\mu)$ is countably-infinite dimensional so any f with discrete spectrum can only have countably many eigenvalues. Thus, by the Luzin-Novikov theorem, there is a Borel function $\psi : U(L^2(\mu)) \times \omega \to \mathbb{C}^\omega$ such that $\{\psi(L, n) : n \in \omega\}$ enumerates the eigenvalues of L. So if we define $\phi : \mathcal{D} \to (S^1)^\omega$ by $\phi(f) = \langle \psi(f, n) \rangle_{n \in \omega}$, this is our desired Borel reduction.

$\square$

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