Complexity of Conformal Equivalence

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1 Introduction

The Riemann Mapping Theorem states that all simply connected domains of $\mathbb{C}$ which are not the whole plane are conformally equivalent. Conformal equivalence is an equivalence relation on the set of all domains of $\mathbb{C}$. Since $\mathbb{C}$ is simply connected, it can only be conformally equivalent to other simply connected domains. So in terms of this equivalence relation, the Riemann Mapping Theorem says that the simply connected domains are split into two equivalence classes: one consisting of just $\mathbb{C}$ and one containing all others.

After the simply connected case, the next type of domains to investigate would be annuli. Since translation is conformal, it suffices to investigate annuli with center 0. If $A_1 = ann(0, r_1, R_1)$ and $A_2 = ann(0, r_2, R_2)$ are such that $\frac{R_1}{r_1} = \frac{R_2}{r_2}$, with $r_1, r_2 > 0$, then $f : A_1 \to A_2$ defined by $f(x) = \frac{r_2}{r_1}x = \frac{R_2}{R_1}x$ is a conformal equivalence. This condition turns out to be necessary as well:

Proposition 1.1 ([6])**.** *Two annuli $A_1 = ann(a, r_1, R_1)$ and $A_2 = ann(b, r_2, R_2)$ with $r_1, r_2 > 0$ are conformally equivalent iff $\frac{R_1}{r_1} = \frac{R_2}{r_2}$.*

But what about the general problem: given an arbitrary domain D, what are the domains conformally equivalent to D? That is, what is the equivalence class of D? For annuli, the equivalence classes are determined by a single real number, the ratio of the outer and inner radii. One can ask if it is possible in general to assign invariants to domains such that two domains are conformally equivalent exactly when they are assigned the same invariants. It is in fact alway possible to assign such invariants; namely, assign each domain its equivalence class. But this is neither effective nor helpful. The framework of Borel reducibility of definable equivalence relations, a part of descriptive set theory, allows one to make sense of this question about assigning invariants. In doing so it also provides a means to compare different types of equivalence problems from across mathematics. We will closely follow the paper [3], also using [2],[4] for background in descriptive set theory and [1],[6] for complex analysis.

2 Descriptive Set Theory and Borel Reducibility

Descriptive set theory is the study of definable subsets of Polish spaces.

Definition 2.1. *A Polish space is a separable, completely metrizable topological space.*

Example 1. *Examples of Polish spaces*

1. *Any countable discrete space.*
2. $\mathbb{C}, \mathbb{C}^n$, *and* $\mathbb{C}^{\mathbb{N}}$ *(the set of complex valued sequences).*
3. $2^{\mathbb{N}}$ *and* $\mathbb{N}^{\mathbb{N}}$ *with the metric* $d(x, y) = 2^{-min\{n: x_n \neq y_n\}}$
4. *For* X *a Polish space, let* $K(X)$ *be the set of compact subsets of* X*. With the topology generated by subbasic open sets of the form* $\{K \in K(X) : K \subseteq U\}$ *and* $\{K \in K(X) : K \cap U \neq \emptyset\}$ *where* U *is open in* X*,* $K(X)$ *is Polish.*

The following definition gives examples of what is meant by 'definable' subsets.

Definition 2.2. *Let* X, Y *be Polish spaces.*

- *A set* $A \subseteq X$ *is Borel if it is in the* σ*-algebra generated by the open sets of* X*.*
- *A function* $f : X \to Y$ *is Borel if the inverse image of every open set is Borel.*
- *A set* $A \subseteq X$ *is analytic if it is the continuous image of a Borel subset of a Polish space.*
- *A set* $A \subseteq X$ *is coanalytic if it is the complement of an analytic set.*

By a theorem of Suslin, a set is Borel iff it is both analytic and coanalytic. There do exist sets which are analytic or coanalytic but not Borel, and of course there are subsets of Polish spaces which are neither analytic nor coanalytic.

Say that an equivalence relation E on a Polish space X is Borel (analytic, coanalytic) if it is Borel (analytic, coanalytic) as a subset of $X \times X$. Such definable equivalence relations are compared to each other via Borel reducibility.

Definition 2.3. *Let E, F be equivalence relations on Polish spaces X, Y. Say that E is Borel reducible to F, written $E \leq_B F$, if there is a Borel function $f : X \to Y$ such that $x_1 E x_2 \iff f(x_1) F f(x_2)$ for all $x_1, x_2 \in X$. Write $E <_B F$ if $E \leq_B F$ but not $F \leq_B E$. Say E, F are Borel bireducible to each other, written $\sim_B$, if $E \leq_B F$ and $F \leq_B E$.*

Example 2. *Examples of definable equivalence relations.*

- *For a Polish space X, $id(X)$ is the identity (or equality) relation on X. If X, Y are any uncountable Polish spaces, then $id(X) \sim_B id(Y)$. The Silver Dichotomy Theorem says that for a coanalytic equivalence relation E on a Polish space X, either E only has countably many equivalence classes, or $id(X) \leq_B E$.*

- *Define E_0 on 2^ω by $x E_0 y \iff \exists m \forall n \geq m[x(n) = y(n)]$. So E_0 is eventual agreement of infinite binary sequences. It is known that $id(2^\omega) <_B E_0$, but in fact much more can be said. The Glimm-Effros dichotomy theorem says that if E if a Borel equivalence relation on a Polish space X, then either $E \leq_B id(2^\omega)$ or $E_0 \leq_B E$.*

- *Let F_2 be the free group on 2 generators and let $\mathcal{P}(F_2)$ be its power set. We can identify $\mathcal{P}(F_2)$ with 2^{F_2} by identifying a subset with its characteristic function. This induces a Polish topology on $\mathcal{P}(F_2)$ with subbasic open sets $\{S : A \subseteq S\}$ for finite $A \subseteq F_2$. With this topology, F_2 acts continuously on $\mathcal{P}(F_2)$ by left translation: $g \cdot A = gA = \{ga : a \in A\}$. This action (and indeed any action) induces an equivalence relation where two elements are related if they are in the same orbit. Let E_∞ be the orbit equivalence relation induced by this action of F_2 on $\mathcal{P}(F_2)$. Let an equivalence relation be called countable if each of its equivalence classes is countable. Then E_∞ is the universal countable equivalence relation; i.e. E_∞ is countable and if E is another countable equivalence relation, then $E \leq_B E_\infty$.*

As a more general setting than just Polish spaces, say $(S, \mathcal{B})$ is a standard Borel space if $\mathcal{B}$ is a σ-algebra on S and there is a Polish topology τ on S which has $\mathcal{B}$ as its Borel sets. Borel reducibility still makes sense if we only assume the spaces involved are standard Borel, since it is the Borel structure and not the particular Polish topology that is important. An important fact is that a Borel subset of a standard Borel space is again standard Borel.

3 Domains and Riemann Surfaces

A domain is simply an open connected subset of $\mathbb{C}$ and a conformal equivalence is an analytic bijection. The definitions of a Riemann surface and conformal equivalences between them are more involved.

Definition 3.1. *A Riemann surface, or a one-dimensional complex manifold, is a pair (X, Φ) where X is a connected, Hausdorff space, Φ is a collection of coordinate patches (U, φ) such that $U \subseteq X$ is open and $\varphi : U \to \mathbb{C}$ is a homeomorphism of U onto $\varphi(U)$, and the following conditions are satisfied:*

1. *Every point of X is covered by at least one patch in Φ.*
2. *If $(U_a, \varphi_a), (U_b, \varphi_b) \in \Phi$ are such that $U_a \cap U_b \neq \emptyset$, then $\varphi_a \circ \varphi_b^{-1}$ is an analytic function from $\varphi_b(U_a \cap U_b)$ to $\varphi_a(U_a \cap U_b)$.*

Simple examples of Riemann surfaces are $\mathbb{C}$ and the Riemann sphere. Also any open connected subset of a Riemann surface is again a Riemann surface; so any domain of $\mathbb{C}$ is another example.

Definition 3.2. *Let $(X, \Phi), (\Omega, \Psi)$ be Riemann surfaces and let $f : X \to \Omega$ be continuous. Let $a \in X$ and $\alpha = f(a)$; then f is analytic at a if for any patch (Λ, ψ) in Ψ containing α, there is a patch (U, φ) in Φ containing a such that*

- $f(U) \subseteq \Lambda$
- $\psi \circ f \circ \varphi^{-1}$ *is analytic on* $\varphi(U) \subseteq \mathbb{C}$.

If f is analytic at each point $a \in X$, then f is analytic on X. If f is a bijection, then f is a conformal equivalence.

To place conformal equivalence in the framework of Borel reducibility, we must realize domains of $\mathbb{C}$ and Riemann surfaces as points in some standard Borel space. This is not too difficult for domains. For X Polish, let $\mathcal{F}(X)$ be the set of closed subsets of X. The Effros Borel structure on $\mathcal{F}(X)$ is the σ-algebra generated by the sets $\{F : F \cap U \neq \emptyset\}$ for $U \subseteq X$ open. With this σ-algebra, $\mathcal{F}(X)$ is a standard Borel space. Similarly let $\mathcal{O}(X)$ be the open subsets of X equipped with the σ-algebra generated by the sets $\{O : U \subseteq O\}$ for $U \subseteq X$ open. Then it is also standard Borel, as it is Borel isomorphic to $\mathcal{F}(X)$ via $\varphi : \mathcal{F}(X) \to \mathcal{O}(X)$ by $\varphi(F) = F^c$. This map is clearly a bijection. It is Borel since

$$\varphi^{-1}(\{O : U \subseteq O\}) = \{O^c : U \subseteq O\} = \{F : U \subseteq F^c\} = \{F : U \cap F = \emptyset\}$$

which is a Borel set in $\mathcal{F}(X)$, and can similarly be shown to have a Borel inverse.

Let $\mathcal{D} \subseteq \mathcal{O}(\mathbb{C})$ be the set of domains in $\mathbb{C}$. This can be shown to be Borel in $\mathcal{O}(\mathbb{C})$, so $\mathcal{D}$ is a standard Borel space. This allows us to realize conformal equivalence of domains as an equivalence relation on a standard Borel space, and thus we can compare it to other equivalence relations using Borel reducibility.

Although it is much more difficult and technical, the same can be done with Riemann surfaces. That is, there exists a standard Borel space $\mathcal{R}$ such that every Riemann surface is conformally equivalent to some element of $\mathcal{R}$. Since every domain is also a Riemann surface, we would like to be able to effectively identify the elements of $\mathcal{R}$ corresponding to the domains in $\mathcal{D}$. And indeed this is the case: there is a Borel function $f : \mathcal{D} \to \mathcal{R}$ such that $d \cong f(d)$ for all $d \in \mathcal{D}$.

Let $\cong_D, \cong_R$ denote conformal equivalence of domains and Riemann surfaces respectively. Then for $d_1, d_2 \in \mathcal{D}$ we have $d_1 \cong_D d_2 \iff f(d_1) \cong_R f(d_2)$. This says that f is a Borel reduction and $(\cong_D) \leq_B (\cong_R)$. This should be expected: determining conformal equivalence of domains is no harder than doing so for Riemann surfaces.

3.1 $Aut(\mathbb{H})$

For a Riemann surface $\mathcal{M}$, an automorphism is a conformal equivalence $\mathcal{M} \to \mathcal{M}$. These form a group under composition, $Aut(\mathcal{M})$. For the half-plane, we have $Aut(\mathbb{H}) = PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{I, -I\}$ where $SL_2(\mathbb{R})$ is the group of 2×2 real matrices with determinant 1, and I is the 2×2 identity matrix. This group acts on $\mathbb{H}$ by Möbius transformations; so

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z) = \frac{az+b}{cz+d}$$

The hyperbolic metric $\rho(x,y) = \ln\left(\frac{|x-\bar{y}|+|x-y|}{|x-\bar{y}|-|x-y|}\right)$ on $\mathbb{H}$ induces the usual topology on $\mathbb{H}$ as a subscpace of $\mathbb{C}$. Two important properties of this action are

- $Aut(\mathbb{H})$ acts on $(\mathbb{H}, \rho)$ by isometries; so for $g \in Aut(\mathbb{H}), \rho(x,y) = \rho(g(x), g(y))$
- Every orbit of this action is discrete.

4 Complexity of Conformal Equivalence

The following theorem holds:

Theorem 4.1. $(\cong_D) \sim_B (\cong_R) \sim_B (E_\infty)$

So conformal equivalence of domains is of the same complexity as conformal equivalence of all Riemann surfaces, and both of these equivalence relations are of the same complexity as the orbit equivalence relation coming from F_2 acting on $\mathcal{P}(F_2)$ by left translation.

From the parametrization we have that $(\cong_D) \leq_B (\cong_R)$. So the theorem follows from proving $(\cong_R) \leq_B (E_\infty)$ and $(E_\infty) \leq_B (\cong_D)$. We will show the second of these. This lower bound for $\cong_D$ is mostly proved by the following proposition:

Proposition 4.2. *Each $A \subset F_2$ can be assigned a discrete set $S_A \subseteq \mathbb{H}$ so the domains $D_A = \mathbb{H} \setminus S_A$ are such that $A\, E_\infty\, B \iff D_A \cong_D D_B$ for $A, B \subseteq F_2$. Furthermore, there is a Borel function $f : \mathcal{P}(F_2) \to \mathbb{H}^{\mathbb{N}}$ such that $f(A)$ enumerates S_A.*

Given the proposition, we need one more result.

Lemma 4.3. *There is a Borel function $g : \mathbb{H}^{\mathbb{N}} \to \mathcal{D}$ such that for $x = \{x_n : n \in \mathbb{N}\} \in \mathbb{H}^{\mathbb{N}}$ discrete, $g(x) = \mathbb{H} \setminus \{x_n : n \in \mathbb{N}\}$.*

Proof. Let $A \subset \mathbb{H}^{\mathbb{N}}$ be the set of sequences enumerating a discrete subset of $\mathbb{H}$. This is a Borel set. Towards defining g, if $x \notin A$, let $g(x) = \mathbb{H}$. If $x \in A$, let $g(x) = \mathbb{H} \setminus \{x_n : n \in \mathbb{N}\}$. This is a Borel function since for any open $U \subset \mathbb{C}$,

$$g^{-1}(\{O : U \subseteq O\}) = \begin{cases} \emptyset & \text{if } U \not\subseteq \mathbb{H} \\ A^c \cup \{\{x_n : n \in \mathbb{N}\} \in A : \forall n\, x_n \notin U\} & \text{otherwise} \end{cases}$$

and hence the inverse image of any subbasic open set is Borel. $\square$

Then given Borel functions f from the proposition and g from the lemma, $h = g \circ f : \mathcal{P}(F_2) \to \mathcal{D}$ is a Borel function such that $h(A) = D_A$. So $A\, E_\infty\, B \iff D_A \cong_D D_B \iff h(A) \cong_D h(B)$. This says $(E_\infty) \leq_B (\cong_D)$.

Proof of proposition 4.2. Recall that $Aut(\mathbb{H}) = PSL_2(\mathbb{R})$. If

$$\sigma = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } \tau \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

then the subgroup $\langle \sigma, \tau \rangle$ in $Aut(\mathbb{H})$ is isomorphic to F_2 and the translation action restricted to this subgroup has no fixed points. We will call this subgroup F_2. Also recall that for $x \in \mathbb{H}$ the orbit of x, $\{g(x) : g \in F_2\}$, is discrete and $g(x) = x$ iff g is the identity of F_2.

Fix $x_1 \in \mathbb{H}$ and for $g \in F_2$ let $x_g = g(x_1)$. So if $h \in F_2$, $g(x_h) = g(h(x_1)) = x_{gh}$. We know $\mathcal{O} = \{g(x_1) : g \in F_2\} = \{x_g : g \in F_2\}$ is discrete, so there is an $\epsilon > 0$ such that $\mathcal{O} \cap B_\epsilon(x_1) = \{x_1\}$. Since F_2 acts by isometries and $g \cdot \mathcal{O} = \mathcal{O}$, we also have $\mathcal{O} \cap B_\epsilon(x_g) = \{x_g\}$ for all $g \in F_2$.

Let $x_1^{(0)} = x_1$ and choose $x_1^{(1)}, x_1^{(2)}, x_1^{(3)}$ from $\mathbb{H}$ so all 4 are distinct, $\rho(x_1^{(0)}, x_1^{(i)}) < \frac{\epsilon}{5}$ for $i = 1, 2, 3$, and $\epsilon_{ij} = \rho(x_1^{(i)}, x_1^{(j)})$ for $0 \leq i \neq j \leq 3$ are distinct (except we must obviously have $\epsilon_{ij} = \epsilon_{ji}$).

Then define $x_g^{(i)} = g(x_1^{(i)})$, so $x_g^{(0)} = x_g$ and $g(x_h^{(i)}) = x_{gh}^{(i)}$. Since acting on $\mathbb{H}$ by $g \in F_2$ is an isometry, $\rho(x_g^{(i)}, x_g^{(j)}) = \rho(x_1^{(i)}, x_1^{(j)})$. For fixed $g \in F_2$ and $0 \leq i \neq j \leq 3$, by the triangle inequality we know $\rho(x_g^{(i)}, x_g^{(j)}) \leq \frac{2\epsilon}{5} \leq \frac{\epsilon}{2}$. Then

$$B_{\frac{\epsilon}{2}}(x_g^{(0)}) \cap \bigcup_{h \in F_2} \{x_h^{(i)} : 0 \leq i \leq 3\} = \{x_g^{(i)} : 0 \leq i \leq 3\}.$$

So since for fixed g the points $x_g^{(i)}$ are some fixed distance that is uniform in g apart from each other, $\{x_g^{(i)} : 0 \leq i \leq 3, g \in F_2\}$ is also a discrete set.

For $A \subseteq F_2$, define $S_A = \{x_g^{(i)} : 0 \leq i \leq 2, g \in F_2\} \cup \{x_h^{(3)} : h \in A\}$. This assignment $A \to S_A$ will satisfy the proposition, and the construction of S_A is explicit enough that there is Borel function $f : \mathcal{P}(F_2) \to \mathbb{H}^{\mathbb{N}}$ so $f(A)$ is an enumeration of S_A. So it remains to show that this assignment works as stated in the proposition.

Suppose $A \, E_\infty \, B$, such that $gA = B$, $g \in F_2$. Then $g(S_A) = \{x_h^{(i)} : h \in F_2, 0 \leq i \leq 2\} \cup \{x_{gh}^{(3)} : h \in A\}$. But this second set is $\{x_r^{(3)} : r \in gA\} = \{x_r^{(3)} : r \in B\}$. So $g(S_A) = S_B$. Since each element of F_2 is an automorphism of $\mathbb{H}$, we also have $g(D_A) = D_B$ and hence $D_A \cong_D D_B$.

In the other direction, assume $\varphi : D_A \to D_B$ is a conformal equivalence. Let $\theta : \mathbb{H} \to \mathbb{D}$ be a conformal equivalence, which exists by the Riemann mapping theorem. Then $\theta \circ \varphi : D_A \to \mathbb{D}$ extends to $\varphi_0^+ : \mathbb{H} \to \mathbb{D}$ by the following:

Lemma 4.4. *[6] Let $G \subseteq \mathbb{C}$ be a domain, $f : G \to \mathbb{D}$ such that $f \in H(G \setminus a)$ for $a \in G$. Then a is removable and f can be extended to be holomorphic on all of G.*

Proof. Assume $G = B_r(a)$. Define h by $h(a) = 0$ and $h(z) = (z-a)^2 f(z)$ for $z \in G \setminus a$. Then

$$\lim_{z \to a} \frac{h(z) - h(a)}{z - a} = \lim_{z \to a} (z-a) f(z) = 0$$

since f is bounded. So $h \in H(G)$ and we can write $h(z) = \sum_{n=2}^{\infty} c_n (z-a)^n$ for $z \in B_r(a)$. Then defining $f(a) = c_2$, we can extend the definition of f to all of G by $f(z) = \sum_{n=0}^{\infty} c_{n+2}(z-a)^n$, showing that $f \in H(G)$. $\square$

Let $\varphi^+ = \theta^{-1} \circ \varphi : \mathbb{H} \to \mathbb{H}$; this is a holomorphic map extending φ. Similarly there is a holomorphic map $\varphi^- : \mathbb{H} \to \mathbb{H}$ extending φ^{-1}. Then φ^+, φ^- are inverses and so φ^+ is actually an automorphism of $\mathbb{H}$ extending φ. We will now refer to this extension as φ. Since φ bijectively maps D_A onto D_B, we also have $\varphi(S_A) = S_B$. Now we just need to show $\varphi = g$ for some $g \in F_2$. Consider $\varphi(x_1^{(0)})$. It must be in S_B, so for some $g \in F_2, 0 \leq i \leq 3$, $\varphi(x_1^{(0)}) = x_g^{(i)}$. It can be shown that in fact $i = 0$, so $\varphi(x_1^{(0)}) = x_g^{(0)}$. Then $\rho(\varphi(x_1^{(1)}), x_g^{(0)}) = \rho(x_1^{(1)}, \varphi^{-1}(x_g^{(0)})) = \rho(x_1^{(1)}, x_1^{(0)}) = \epsilon_{10}$. Since all the ϵ_{ij} were distinct, it must be that $\varphi(x_1^{(1)}) = x_g^{(1)}$. Similarly $\varphi(x_1^{(2)}) = x_g^{(2)}$. Then $\varphi(x_1^{(i)}) = x_g^{(i)}$ for $0 \leq i \leq 2$ and since φ and g are Möbius transformations agreeing at 3 points, $\varphi = g$ by Proposition 3.3.9 in [1].

The last thing we must do it to check that $gA = B$. If $h \in A$, then $x_h^{(3)} \in S_A$, so $g(x_h^{(3)}) = \varphi(x_h^{(3)}) = x_{gh}^{(3)} \in S_B$. So $gh \in B$. The reverse containment is symmetric, and the proof is completed.

$\square$

As a final remark, in obtaining the lower bound, we used domains of the form $\mathbb{H} \setminus S$ where S is a discrete set. We could have used $\mathbb{D} \setminus S$ since $\mathbb{H}$ and $\mathbb{D}$ are conformally equivalent. In either case, the result shows that E_∞ is Borel equivalent to conformal equivalence restricted to domains of the form $\mathbb{H} \setminus S$. So the problem of conformal equivalence restricted to sets obtained by removing an infinite discrete set from $\mathbb{H}$ or $\mathbb{D}$ is already as complicated as the general problem for domains or for Riemann surfaces.

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